Continuity and discontinuity

Note Title 24/09/2008

This lecture looks at a few "extra" aspects of continuity.

- Why is discontinuity a bad thing for numerical computation?
- Do continuous functions exist in the real world?
- Do discontinuous functions?
- Are inverse functions continuous?

On the scale at which we live and perceive the world, continuity seems a good model: pressure is a continuous function of depth in the sea, the velocity of a smoothly-driven car is a continuous function of acceleration (and even a violent crash looks continuous on a slow-motion film). Movies look continuous, for that matter.

On closer inspection, it's not so simple. Pressure results from the individual impacts of innumerable water molecules (and air molecules above those). And movies are just lots of still photos taken in rapid succession and displayed the same way, fooling the eye with the illusion of continuous motion.

Stage magicians (at least some of them) can move faster than one can see, almost discontinuously. The justly famous Bruce Lee could punch so fast his motion was invisible on camera: they had to use slow motion to make him look just "fast".

So the notion of physical continuity is elusive. Nonetheless, it's a good <u>model</u> for what we see, and gives good results in many branches of science and engineering. So it's worthwhile exploring the mathematical idea of 'perfect' continuity. Similarly, use of mathematical infinity makes sense as a first approximation to "humongously large" which is more often the actual case.

On a computer, discontinuous functions are a problem. Consider

$$P(x) = \begin{cases} 0 & \text{if } x < \pi \\ 1/2 & \text{if } x = \pi \\ 1 & \text{if } x > \pi \end{cases}$$

Input : $x = 3.14159265358979 32384$

what will the output be?

(strictly speaking, this is $< \pi$ so it ought to be 0).

Answer: it depends on the details of how input is handled! If it's rounded automatically (on conversion to binary or hex, for example) it might go to 3.1416, maybe; and return $\underline{1}$. or to 3.14159265 and return 0.

So, this discontinuous function is hard to compute. Contrast

$$S(x) = \begin{cases} 0 & \text{if } x < \pi \\ x-\pi & \text{if } x \geq \pi \end{cases}$$

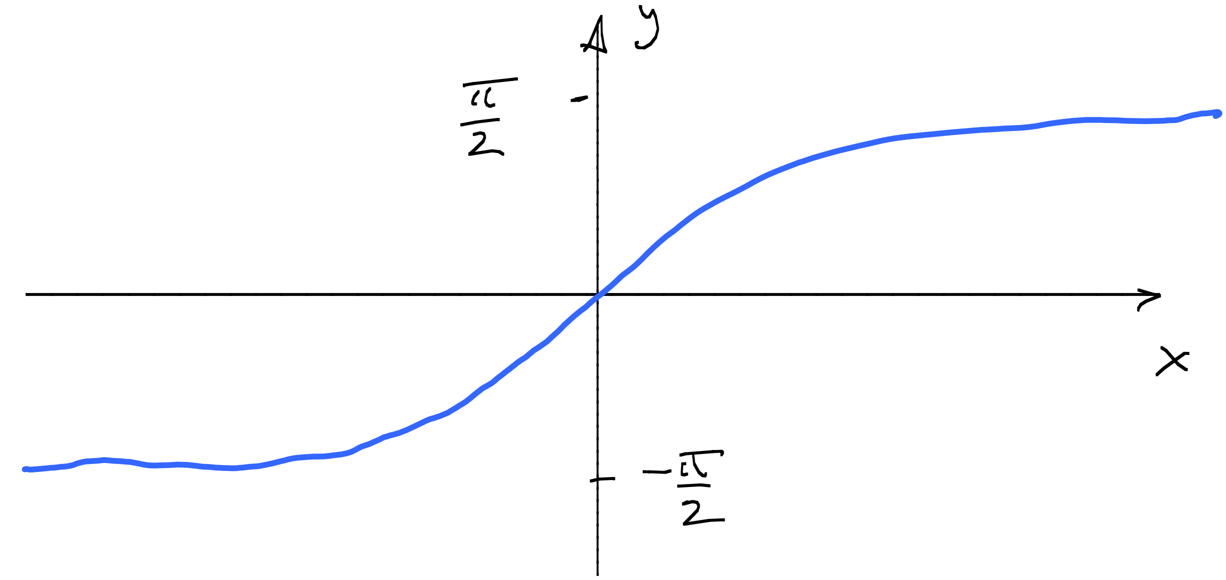
which is continuous: input of a number near π will return an answer near 0, no matter which branch is used, and no matter what (small) rounding happens.

Inverse Functions

I said last lecture that if $f(x)$ is cts, so is $f^{-1}(x)$; but one needs to watch the edges.

Example $\quad y = \arctan x$
$\qquad\qquad\quad = \tan^{-1} x$

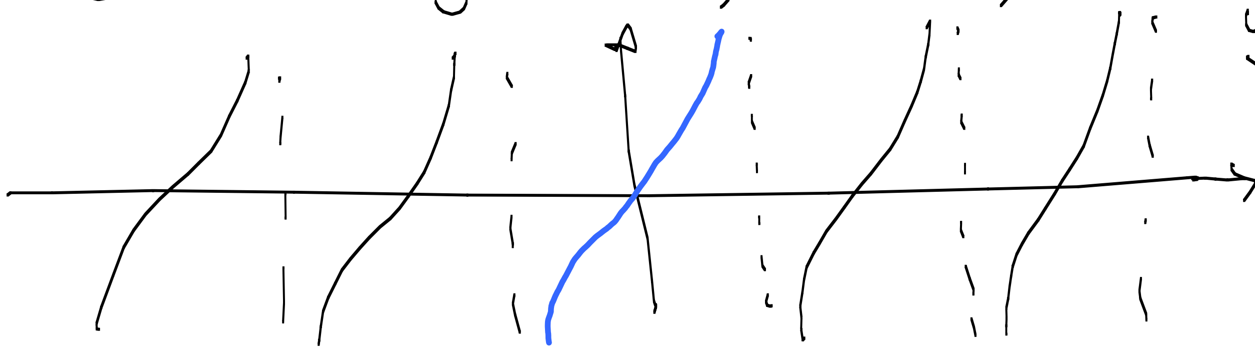
:- $\tan y = x$



obviously continuous (for all x).
Don't forget the restriction

$$-\frac{\pi}{2} < y < \frac{\pi}{2} \quad !$$

'its inverse function, $\tan x$, is cts



ON THAT BRANCH

but only on one branch!
(So you need to be alert).

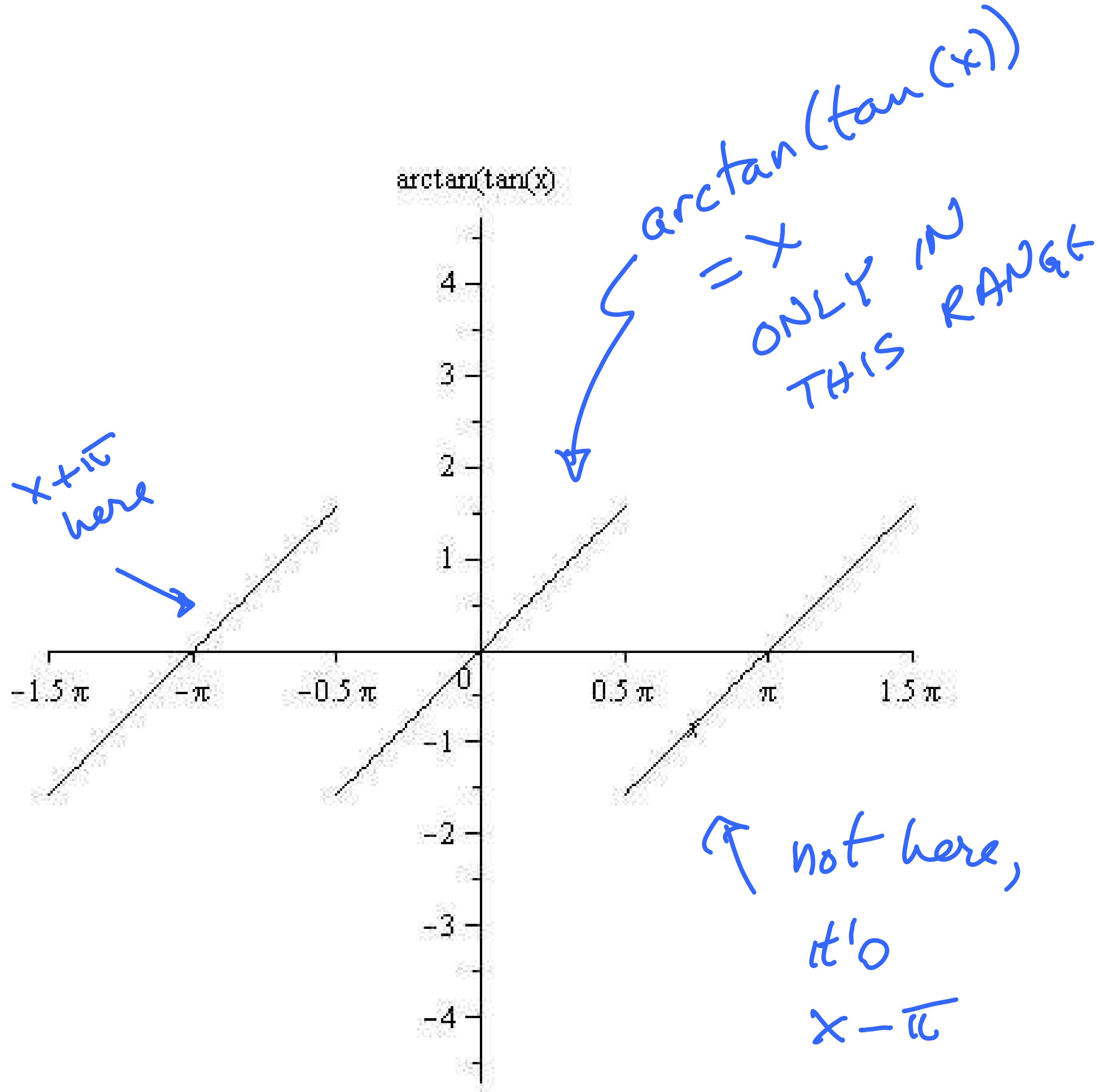
Tricky examples:

$$y = \tan \tan^{-1} x$$

$$y = \tan^{-1} \tan x$$

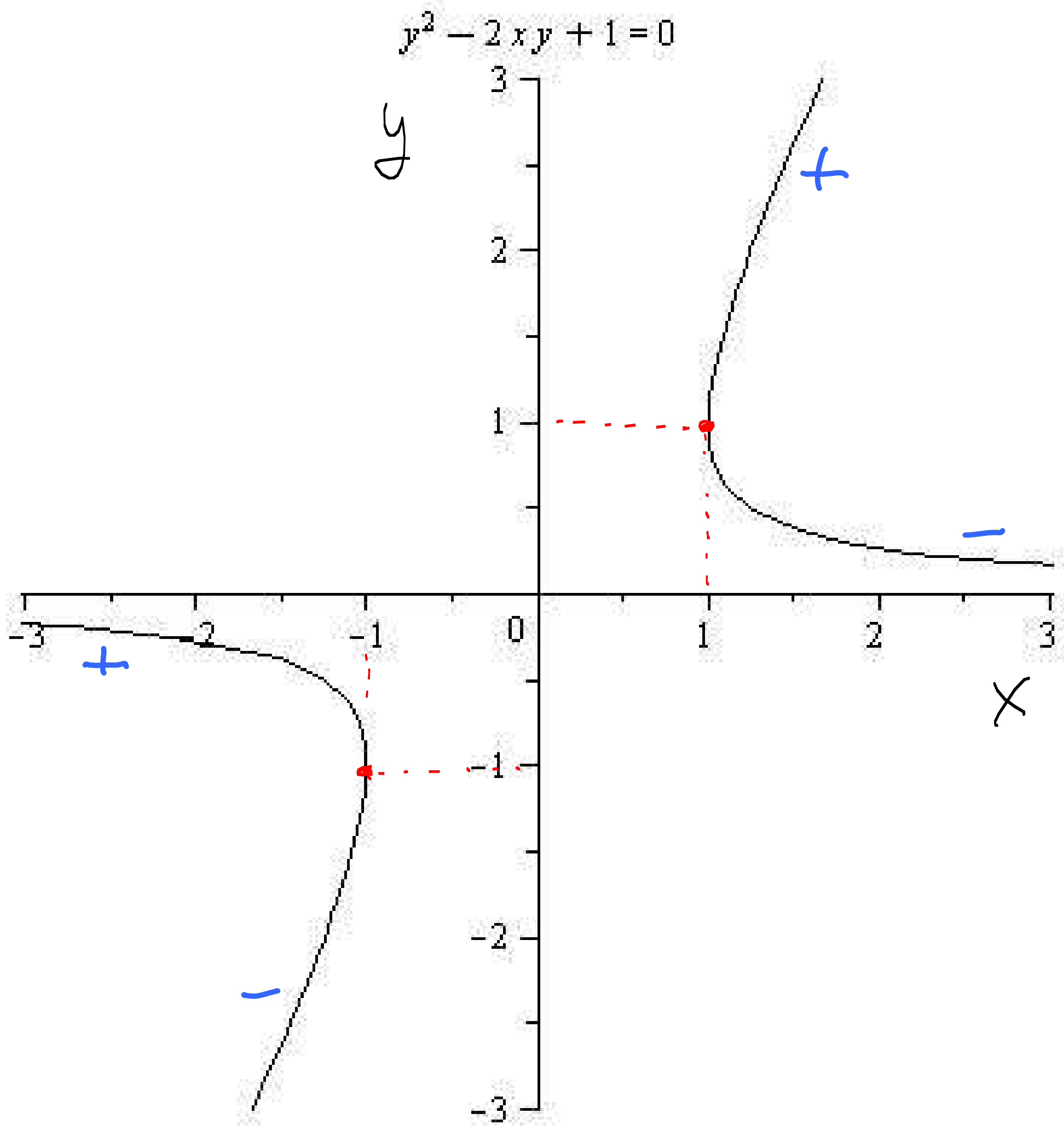
The first is the same as $y = x$ for all x. $\tan^{-1} x$ is valid for all x, and it is "the angle whose tangent is x". Taking its tangent gives you x.

The second is trickier! Plot it, to see:



Ouch! the discontinuities in tan(x) have mucked it up!

Other inverse functions



Plotted by isolating x

$$y^2 + 1 = 2xy$$

$$\frac{y^2+1}{2y} = x \qquad \text{plot } x \text{ vs } y.$$

Of course we may solve this equation for y as a function of x explicitly:

$$ax^2 + bx + c = 0$$

$$\Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

here

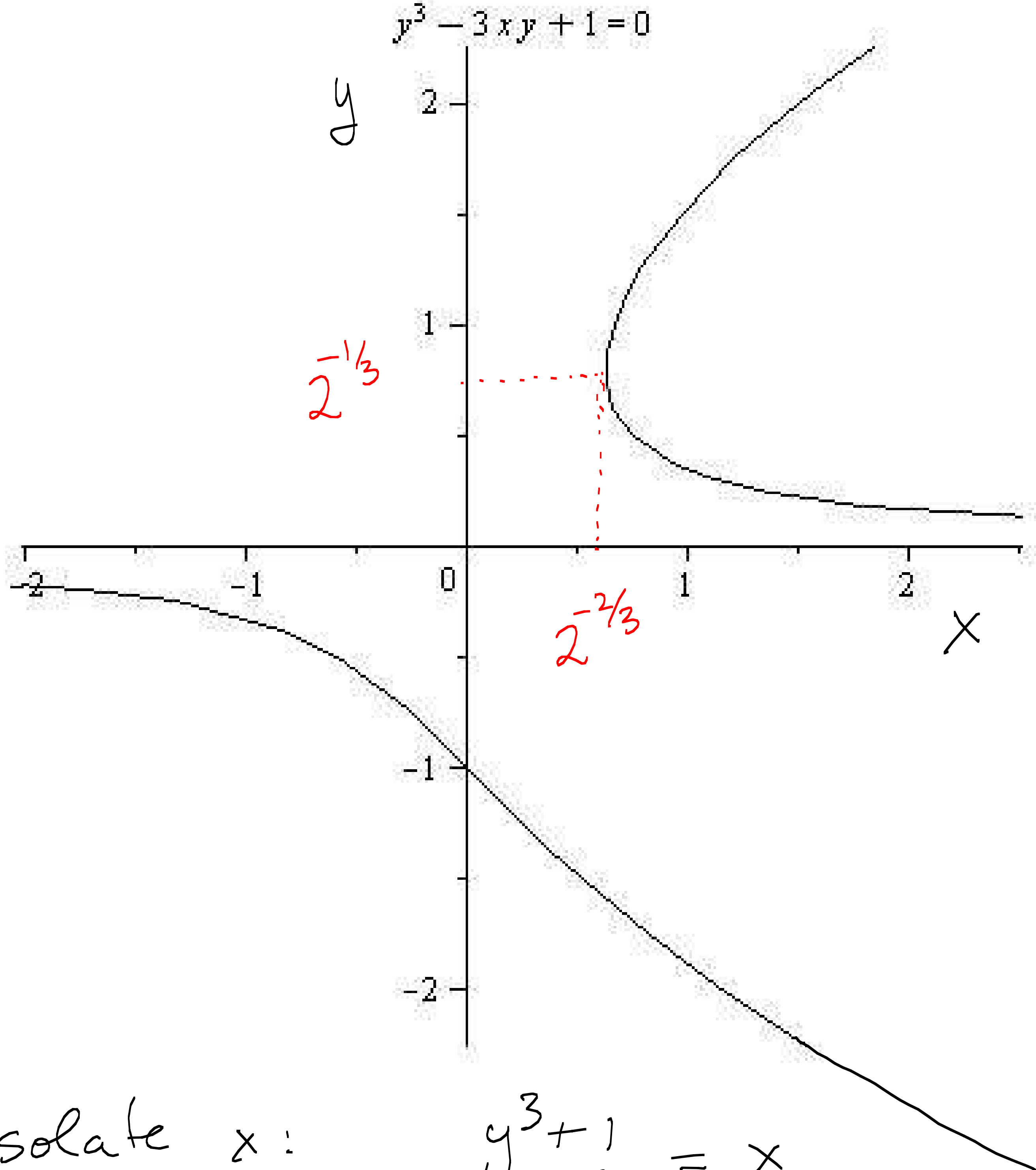
$$y^2 - 2xy + 1 = 0$$

$$\Rightarrow \quad y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$= x \pm \sqrt{x^2 - 1}$$

So there are <u>two</u> functions with (real) domain $|x| \geq 1$.
Each is cts, when restricted.

Similarly $y^3 - 3xy + 1 = 0$



Isolate x: $\dfrac{y^3+1}{3y} = x$

and plot x vs y.

We can solve this one too, but even without doing so, we can see that, away from where the range of y must be restricted,

y is continuous as a function of x because x is a continuous function of y.

We are very close to the "Implicit Function Theorem" (but we'll wait till we've done derivatives, so the red points can be identified).

Extra extra: How to solve cubics

Main trick: once there is no y^2 term, split the unknown:

put $y = u + v$.

Here $$y^3 - 3xy + 1 = 0$$

$$(u+v)^3 - 3x(u+v) + 1 = 0$$

$$u^3 + \underbrace{3u^2v + 3uv^2}_{\text{keep together, \& factor}} + v^3 - 3x(u+v) + 1 = 0$$

$$u^3 + \underbrace{3uv(u+v) + v^3 - 3x(u+v)}_{\text{now combine}} + 1 = 0$$

$$u^3 + v^3 + 1 \quad + (3uv - 3x)(u+v) = 0$$

If we can solve these separately then we're done:

$$u^3 + v^3 + 1 = 0$$

$$(3uv - 3x)(u+v) = 0$$

$y = u + v = 0$ is <u>not</u> a solution

$y^3 - 3xy + 1 = 0$

so we may divide by $u+v$

$$u^3 + v^3 = -1$$

$$uv = x$$

This pair of eq'ns is easy to solve

$$(u^3 + v^3)^2 = (-1)^2 = 1$$

$$u^6 + 2u^3v^3 + v^6 = 1$$

at the same time $4u^3v^3 = 4x^3$

$$\therefore \quad u^6 - 2u^3v^3 + v^6 = 1 - 4x^3$$

so $(u^3 - v^3)^2 = 1 - 4x^3$

or $u^3 - v^3 = \pm\sqrt{1-4x^3}$

remember $u^3 + v^3 = -1$

so

$$2u^3 = -1 + \sqrt{1-4x^3}$$

$$2v^3 = -1 - \sqrt{1-4x^3}$$

(using the opposite signs on $\sqrt{\ }$ just reverses u & v, which makes no difference to $y = u+v$)

Take $u = \left(\dfrac{-1+\sqrt{1-4x^3}}{2}\right)^{1/3}$

(there are 3 complex u)

and use $uv = x$ to get

$$v = \frac{x}{u}$$

so $y = u + \dfrac{x}{u}$

for each of the three u.

Explicitly,

$$y = \left(\frac{-1 + \sqrt{1-4x^3}}{2}\right)^{1/3} + \frac{x}{\left(\frac{-1 + \sqrt{1-4x^3}}{2}\right)^{1/3}}$$

Three values of y, one for each cube root.

<u>nb</u> remember $z^{1/3} = r^{1/3} \angle \theta/3$

and $r^{1/3} \angle \frac{\theta}{3} + \frac{2\pi}{3}$ and $r^{1/3} \angle \frac{\theta}{3} + \frac{4\pi}{3}$.

Bonus question: what is

$$\lim_{x \to 0} \frac{x}{\left(\frac{-1+\sqrt{1-4x^3}}{2}\right)^{1/3}}$$

(for each cube root?)

Is this formula continuous at $x=0$?

$$\cancel{\infty}$$